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U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS
CENTRAL RADIO PROPAGATION LABORATORY
WASHINGTON, D. C.

# AVERAGE RADIO RAY REFRACTION IN THE LOWER ATMOSPHERE

BY M. SCHULKIN

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# AVERAGE RADIO RAY REFRACTION IN THE LOWER ATMOSPHERE*

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Abstract:

The average refractive bending of radio-frequency rays has been calculated from actual radio-frequency refractive index distributions with height as computed from mean radiosonde data. The maximum possible bending occurs for rays passing entirely through the atmosphere and arriving or departing tangentially at the earth's surface. The range of this total angular ray bending extends from 11 milliradians (mr) (0.63°) at Fairbanks, Alaska, in April, to 18 mr (1.01°) at San Juan, Puerto Rico, in July, and is about 14 mr (0.80°) at Washington, D.C., in October. About 90% of the ray bending occurs in the lowest 10 km of the atmosphere. These results are compared with the ray bending computed from the 4/3 effective earth's radius approximation.

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* This paper is the first part of a standard refraction study, the goal of which is the calculation of field intensity as affected by average refraction conditions when one or both of the terminals is very high, such as a high-flying aircraft or rocket. By the "lower atmosphere" is arbitrarily meant that portion of the atmosphere up to about 18 km. However, an estimate is made in this study of the contribution to the refraction of the atmosphere above this elevation, excluding any ionospheric effects.

## 1. Introduction

The problem of radio-ray tracing is closely connected with the problem of astronomical refraction. The principal difference between optical-ray and radio-ray refraction is due to the effect of the large contribution of water vapor to the refractive index of the atmosphere at radio frequencies. The history of atmospheric optical-ray tracing is a long one and is mostly devoted to the computation of astronomical refraction and somewhat to geodetic surveying. Garfinkel(1) reviews the past history of astronomical refraction calculations, and he presents a method of obtaining total optical refraction for a ray passing entirely through the earth's atmosphere for angles of elevation less than $10^{\circ}$. In all the astronomical refraction investigations, the air density-height distribution function is approximated by plausible physical models. Garfinkel fits mean sounding data to his model, which consists of a temperature lapse up to the stratosphere, and then an isothermal stratosphere. He shows that his results agree quite well, within the error of measurement, with the semi-empirical tables of Radau and Pulkowa.

The tabulated sounding data which he used were taken from Humphreys(2) and were representative of four European cities between the years 1900 and 1912. Since the date of Garfinkel's paper (February 1944), the United States Weather Bureau has published (May 1945) averages of radiosonde data taken for the most part from July 1939 to December 1943 for the North American network of stations from Alaska to the Caribbean Sea.(3) These data are used in the present computations.

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(1) "An Investigation in the Theory of Astronomical Refraction," by B. Garfinkel, Astronomical Journal, Vol. 50, No. 8 (2/44).

(2) "Physics of the Air," by W. J. Humphreys, McGraw-Hill (1929) pp. 54-56.

(3) "Upper Air Average Values of Temperature, Pressure, and Relative Humidity over the United States and Alaska," by R. Ratner, Climate and Crop Weather Div., U.S.W.B., Washington, D.C. (5/45).

2. Atmospheric Refraction* of Radio Waves.

The object of atmospheric radio wave refraction studies usually is to find a convenient method of correcting the radio wave field-intensity equations for this effect. This final step has not been taken in this paper, but a method of obtaining the true refraction is presented in Section 4, page 8. A survey of existing methods is made herein in terms of refraction calculations, instead of field-intensity applications for which they were originally designed, and it is shown how widely divergent the results of the different methods are from the true refraction. We show that the problem is one of correcting for refraction at different heights in the atmosphere and for different seasons at different geographical locations. The most widely used method of approximation is obtained by using an effective earth's radius equal to 4/3 the actual radius. We furnish data to examine the adequacy of the 4/3 earth's radius approximation.

In the present paper, refraction has been computed from the actual refractive index structure of the atmosphere as obtained from average meteorological data at the location in question. The assumption is made, as is usually done, that the atmosphere is horizontally homogeneous over that location. Calculations have been made for different refractive index structures over different locations. The results have been plotted as the solid curves A, B, C in Figure 1, page 8. Curves A, B, and C represent the actual variation of refraction with height, computed by methods presented in this paper, for Fairbanks, Alaska, in April, for Washington, D.C., in October, and for San Juan, Puerto Rico, in July. Broken curves A', B', and C' represent the approximations obtained by a method due to Pearcey (see below) using a 4/3 earth's radius factor up to the height at which the refractive index becomes unity and considering no refraction beyond this point.

The earliest successful attack on the problem of correcting radio propagation analysis for atmospheric refraction was introduced by Schelleng, Burrows, and Ferrell[4] when they introduced

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* Refraction is defined here as the total bending of a ray over a path. The curvature at a point on a curve is defined as the rate of change of the tangent direction per unit length of curve at that point, and is measured in terms of change of angle per unit length of path. For a circle, the curvature at any point is constant and is equal to the reciprocal of the radius. When the curvature is known along a ray, the refraction may be computed as the integral of the curvature over the path.

(4) "Ultra Short Wave Propagation," by J. C. Schelleng, C. R. Burrows, and E. B. Ferrell, Proc. I.R.E., Vol. 21, p. 427 (3/33).

the concept of effective earth's radius. They found that a factor of 4/3 was satisfactory for annual average conditions, up to 0.5 km as deduced from Humphreys'[2] weather data. However, the correction thus introduced is only strictly correct for an atmosphere with linear refractive index gradient of $39 \times 10^{-6}$ units per km, and thus cannot possibly hold beyond the height (approximately 10 km) at which the refractive index has reached the value of unity. In addition, the atmosphere has a non-linear refractive index gradient, and the true correction will vary with elevation and distance.

Eckart and Plendl[5] showed how to correct for atmospheric refraction for nearly horizontal rays by fitting the actual refractive index structure by means of two parabolas, one from the surface to 5 km, and the other from 5 to 10 km. Above that elevation they considered the refraction negligible. The method furnished satisfactory tables of corrections for a mean atmosphere to 10 km. To extend the corrections to greater heights or to different refractive index distributions as encountered in different geographical locations would require additional parabolic fits of data and extensive calculations.

Stickland[6,7] attempted to correct radio field-intensity calculations for the variation of refraction with height. An empirical relation was found for the average atmospheric refractive index distribution in terms of the ratio m of the ray radius of curvature at any height h in the atmosphere for nearly horizontal rays,

$$m = 2.5 + 1.0\ h \tag{1}$$

h being given in km. Meteorological conditions in the lowest kilometer are admittedly quite variable, but the formula was considered valid to 10 km, above which height the refraction was assumed to have ceased. It was then suggested that for computational purposes, an average ratio $\bar{m}$ be used between any two points close together at heights $h_1$ and $h_2$, $\bar{m} = 2.5 + 0.5(h_1 + h_2)$. When one point is near the ground, $h_1 \cong 0$, and then we have:

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(5) "Die Uberwindung der Erdkrummung bei Ultrakurzwellen durch die Strahlenbrechung in der Atmosphare," by G. Eckart and H. Plendl, Hochfreq. und Elektroakustik, Vol. 52, No. 2 (8/38) pp.44-58.

(6) "The Effect of Atmospheric Refraction on the Propagation of Radio Waves," by A. C. Stickland, R.R.B./S. 10 (3/43).

(7) "The Dielectric Constant of Water Vapour and its Effects on the Propagation of Very Short Waves," by A. C. Stickland, R.R.B./S.2 (5/42).

$$\bar{m} = 2.5 + 0.5\ h \qquad (2)$$

If we use this average $\bar{m}$ for calculating the actual refraction of a tangential ray, we get incorrect results. In order to demonstrate this, we will now proceed to work out the total refraction in terms of Stickland's average m concept, and show that these values disagree with the true values of refraction as computed by the method outlined later on in this report. The Stickland approximation calls for the use of a single average value of $\bar{m}$ between the ground and the height h and implies that we may use for this layer that value of effective earth's radius factor $\bar{k}$, related to $\bar{m}$ in equation 2 by:

(earth's curvature) - (ray curvature) = ("effective" earth's curvature)

$$\frac{1}{a} - \frac{1}{\bar{m}a} = \frac{1}{\bar{k}a}$$

or: $$\bar{k} = \frac{\bar{m}}{\bar{m}-1} \qquad (3)$$

Now a general expression for the refraction $\tau$ for a tangential ray through a given layer of depth h, with an effective earth's radius factor k corresponding to some linear gradient of refractive index is:

$$\tau = \sqrt{\frac{2h}{a}}\left(\sqrt{k} - \frac{1}{\sqrt{k}}\right)$$ (See Appendix 4, page 20)

where: $\tau$ is the refraction in radians.
a is the earth's radius, expressed in the same units as h.
h is the depth of the layer, measured from the ground.
k is the effective earth's radius factor.

The bending of a radio ray between the ground and height h, obtained from the last equation by using the values of k obtained from equations (2) and (3) has been plotted in Figure 2, page 24, as Curve A. In this same figure we have also plotted: (1) Curve B, the actual refraction for a Stickland atmosphere defined by equation (1) and computed by the method presented in this report, (2) the mean refraction for Washington, D.C., in October, and (3) the 4/3 earth's radius approximation. It may be seen that the Stickland averaging process gives too low values of refraction, and that the simple 4/3 earth's radius approximation is at least as satisfactory as Stickland's refraction correction. The reason for the failure of Stickland's formula in this case is that the method of averaging m has the effect of weighting all height intervals equally, whereas the concept of the refraction as the integrated curvature along the ray path allows for the condition that almost horizontal rays have much longer paths in the lower layers where the refraction is greatest.

Pearcey[8] has given a simple arithmetical formula (equation 4 below) for correcting for the refraction of solar radio rays arriving at low angles $\theta_o$, a procedure necessary in solar noise studies, by using the surface value of the refractive index and then assuming a linear refractive index gradient of 39 x $10^{-6}$ units/km up to that height at which the refractive index becomes unity. Above this height the refraction is considered to have ceased. This is equivalent to using a 4/3 earth's radius factor but, in addition, the final height is specified at which the correction is applicable. This model has the advantage of being simple and the validity of the approximation for total bending may be examined in the dashed curvesof Figure 1, page 23. Pearcey's formula for low ground angles of arrival or departure, $0 \leqq \theta_o \leqq 10^o$ is given by:

$$\tau = \frac{\Delta n_o}{\left(\frac{h_2}{a} - \Delta n_o\right)} \left[\sqrt{2\left(\frac{h_2}{a} - \Delta n_o\right) + \theta_o^2} - \theta_o\right] \text{ in radians} \qquad (4)$$

where: $\tau$ is the total bending in radians.
$\Delta n_o$ is $(n_o-1)$ where $n_o$ is the value of the refractive index at the earth's surface.
$h_2$ is the elevation at which $\Delta n$ becomes zero under a standard refractive index gradient of 39 x $10^{-6}$ units/km.
a is the earth's radius, expressed in the same units as $h_2$.

From Figures 1 and 2, it may be seen that the refraction approximations in use at the present time are not quite adequate for all seasons, geographical locations or at very high altitudes. However, the effect on field-intensity calculations of the deviations of the refraction approximations from the true refraction has not been determined in this study. It is contemplated to study these effects and to present the results in a subsequent paper. It is also intended to use the receivers of existing radar installations to get an experimental measure of the total atmospheric refraction of radio-frequency noise from the sun at one or two frequencies between 100 Mc/s and 10,000 Mc/s for comparison with the refraction computed from nearly simultaneous meteorological soundings.

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(8) "Solar Radiation at Radio Frequencies and its Relation to Sunspots," by J. L. Pawsey, R. Payne Scott, L. L. McCready, R.P.R. 24 (6/46) Commonwealth of Australia, Council for Scientific and Industrial Research, Division of Radiophysics.

## 3. Meteorological Data and Refractive Index Computations.

The formula for the radio-frequency* refractive index of moist air is[9]:

$$(n-1) \times 10^6 = \frac{79p}{T} \left(1 + 4800 \frac{e}{pT}\right) \tag{5}$$

Equation (5) may be written in the form:

| | Optical and Radio Frequency Term | | Explicit Water Vapor Term Required only at Radio Frequencies | |
|---|---|---|---|---|
| $(n-1) \times 10^6 =$ | $79p/T$ | $+$ | $3.79 \times 10^4 \; e/T^2$ | (5a) |

where: n is the refractive index.
p is the total pressure in millibars (mb) and is equal to $p_d + e$.
$P_d$ is the partial pressure of dry air.
e is the water vapor pressure.
T is the temperature in °K, and is equal to 273 + t° C.

This is a semi-empirical formula and was obtained as a result of radio wave propagation studies and laboratory measurements of the refractive index of water vapor and dry air. The adequacy of equation (5) as a basis for refraction calculations is discussed in Appendix 1, page 12. Charts based on this formula have been constructed for obtaining the refractive index of moist air at radio frequencies from meteorological data to heights of 25 km. In this report, representative radiosonde data published by the U.S. Weather Bureau[3] have been used for the calculation of the refractive index distribution for three North American stations separated widely in geographical location for one month of each season. The computations were made by

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* The formula is applicable for frequencies at least up to 30,000 Mc/s. Although some variation of refraction might be expected to exist near various absorption lines associated with various elements in the atmosphere, Van Vleck[10] has pointed out that the observed magnitude of the absorption coefficient of oxygen and water vapor in the atmosphere whose absorption lines lie below 30,000 Mc/s would not be expected to be accompanied by appreciable changes in refraction. Further investigation may show that formula 5 is applicable at frequencies even considerably in excess of 30,000 Mc/s.

(9) "Nomograms for Computation of Modified Index of Refraction," by R. A. Burgoyne, M.I.T. RL #551 (4/45).

(10) "The Relation between Absorption and the Frequency Dependence of Refraction," by J. H. Van Vleck, M.I.T. RL #735 (5/45)

drawing graphs respectively for the contribution due to the total pressure term, 79 p/T, important both at optical and radio frequencies, and that due to the explicit water vapor pressure term, $3.79 \times 10^4$ e/$T^2$, important only at radio frequencies. Figures 3-6, pages 25-28 give the contribution of the total pressure term to the refractive index computation. Figures 7 and 8, pages 29-30 , give, in terms of relative humidity(RH) and temperature, the contribution due to the explicit water vapor term (Relative Humidity-Temperature Term). The relation between e and RH is $e = e_s$ (RH)/100, $e_s$ being the saturation water vapor pressure which is only a function of temperature. The explicit water vapor term then becomes $3.79 \times 10^2$ $e_s$ (RH)/$T^2$.

Data on geographical and seasonal distributions of refractive index have been examined, and it is found that the extremes are represented by Fairbanks, Alaska, in April and by San Juan, Puerto Rico, in July. The data for Washington, D.C. in October were found to be approximately a mean between these two extremes.

Ordinarily soundings go up only to 15 to 18 km. However, at these elevations the atmosphere is approximately isothermal (stratosphere), and the refractive index distribution may be easily extended to great heights, at least to 30 km, which is as far as the isothermal condition prevails(11). (See Appendix 2, page 16).

There are limitations on the accuracy of both the refractive index formula, equation (5a), and the meteorological observations in an individual sounding. These limitations are discussed in Appendix 1, page 12, where there is shown to be an uncertainty in the refractive index calculations of the order of 7%, about half being due to the uncertainty in the formula coefficients, and half due to the errors of meteorological measurement, the humidity measurements being the least reliable.

4. Computation of Refraction.

A method is now presented for computing the refraction of radio-frequency rays through an atmosphere of known refractive index distribution, under the usual assumption that surfaces of equal index of refraction are spherical and concentric with the earth (horizontal homogeneity). The refraction $\tau_{1,2}$ of a ray, as illustrated in Figure 9, page 31, between two heights $h_1$ and $h_2$ through an atmosphere of known refractive index distribution and making an angle θ with the spherical refracting surface at the height h is derived in Appendix 3, page 17, as:

$$\tau_{1,2} = \overline{(\cot \theta)}_{1,2} \, ( \Delta n_1 - \Delta n_2) \qquad \text{in radians} \qquad (6)$$

(11) "Tentative Tables for the Properties of the Upper Atmosphere". - C. N. Warfield, N.A.C.A. Technical Note 1200 (1/47).

where, $\Delta n$ is $(n-1)$ and $\overline{(\cot \theta)}_{1,2}$ is the mean value of the cotangent function over the interval $h_1$, $h_2$. Our problem is solved when we are able to obtain a proper value of $\overline{(\cot \theta)}_{1,2}$, the complicating factors being that $\theta$ is a function of both $\Delta n$ and h as determined from Snell's law for concentric spherical refracting surfaces, and further that $\Delta n$ is only known empirically as a function of h, being given by a table of measured values as obtained from radiosonde data. Thus according to Snell's law, the angles $\theta$ for a ray path through concentric spherical refracting surfaces are determined from the formula,(12)

$$nr \cos \theta = n_o r_o \cos \theta_o = \text{constant} \tag{7}$$

where: $r = a + h$, is the radius of the spherical refracting surface, at the elevation h, and a is the earth's radius, and where the subscript zero refers to the ground level.

At small elevation angles, $\theta_o \leqq 10^o$, equation (7) transforms, neglecting quantities of second and higher orders, to(12):

$$\frac{\theta^2 - \theta_o^2}{2} = \left( \Delta n + \frac{h}{a}\right) - \left( \Delta n_o + \frac{h_o}{a}\right) \tag{8}$$

and we then have, solving for $\theta$,

$$\theta = \sqrt{\theta_o^2 + 2\left( \Delta n + \frac{h}{a}\right) - 2\left( \Delta n_o + \frac{h_o}{a} \right)} \tag{9}$$

where $h_o$ is the station elevation above mean sea level. This enables us to calculate $\theta$ at any elevation, provided that the calculated value of $\theta$ is not appreciably greater than 10°, since we are given for each station a table of values of $\Delta n$ against h. The $\Delta n$ data come from the refractive index distribution with height, which are obtained from the observed meteorological data, as discussed in Section 3. It is necessary to take into account several height intervals, over each of which $\overline{(\cot \theta)}$ may be conveniently determined in the application of equation (6). It is shown in Appendix 4, page 19, how the number and size of these height intervals is determined. Only four height intervals are required from the ground to 18 km, namely, (1) surface to 0.5 km; (2) 0.5 to 2.5 km; (3) 2.5 to 6.0 km; (4) 6 to 18 km. Above 18 km the contribution to the total refraction is small and may be shown to be given approximately by $\tau = \Delta n_{18} \cot \theta_{18}$ where the subscripts refer to the values of the quantities at 18 km. Over these intervals at these low angles, it may be shown that $\overline{(\cot \theta)} \cong 1/\theta_m$, where $\theta_m$ is the arithmetical

(12) "Tropospheric Propagation and Radio-Meteorology," WPG 5, Reprint CRPL-T3 (10/46).

mean of $\theta$ on the interval. Thus, for the interval $(h_1, h_2)$, we have $\theta_m = (\theta_1 + \theta_2)/2$. When we substitute this approximation in equation (6), with $\theta_m$ in radians, we obtain simply,

$$\tau_{1,2} = \frac{\Delta n_1 - \Delta n_2}{\theta_m} \quad \text{in radians} \quad (0 \leqq \theta_o \leqq 10^o) \tag{10}$$

where the angles $\theta$ are determined from equation (9) above, and the $\Delta n$ are obtained from meteorological data as described in Section 3.

Whenever the angle of arrival or departure $\theta_o$ is greater than $10^o$, the total refraction through the entire atmosphere is very small and can be obtained within an error of 4% from the simple formula derived from equation (6) in Appendix 4, page 19,

$$\tau = \Delta n_o \cot \theta_o \quad \text{in radians} \quad (10^o < \theta_o \leqq 90^o) \tag{11}$$

the approximation implied being that $\overline{\cot \theta} \cong \cot \theta_o$.

The mean ray bending for radio-frequency rays over the range of angles of arrival or departure has been obtained for San Juan, Puerto Rico (July) and Fairbanks, Alaska (April), as examples of extremes; and for Washington, D.C. (October) as a mean case. These data have been plotted in Figures 10 and 10a, pages 32-33. In these same figures, we have plotted our computed values of optical refraction for Washington, D.C., in October and compared them with the mean observed astronomical refraction data[13] as adjusted to these same surface refraction conditions. Optical refraction for Washington, D.C. in October was computed by the methods of this section, using the optical refractive index distribution obtained by neglecting the explicit water vapor term in the radio-frequency refractive index computation. Good agreement is obtained which provides a check of the computational procedure.

It is now possible to determine the ray path as a function of the distance d along the earth's surface as illustrated in Figure 9, page 31. $\theta$ has been determined from equations 7 or 9, and $\tau_{1,2}$ has been determined from equations 10 or 11 and it may be shown from geometric considerations in Figure 9 that:

$$\tau_{1,2} = \frac{d_{1,2}}{a} + (\theta_1 - \theta_2) \quad \text{in radians} \tag{12}$$

and we then have: $$d_{1,2} = a\,(\tau_{1,2} + \theta_2 - \theta_1) \tag{13}$$

where: $\tau_{1,2}$, $\theta_2$ and $\theta_1$ are expressed in radians, and d is expressed in the same unit as a, the earth's radius.

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(13) "Astronomy," Vol. 1--Solar System by H. N. Russell, R. S. Dugan and J. Q. Stewart, Ginn, 1945.

5. Conclusion

It has been found that the actual air refraction in the lower atmosphere deviates considerably from that which has been assumed when we use an effective radius of the earth equal to 4/3 of its actual value as an allowance for this refraction. Thus, although the 4/3 earth's radius assumption provides a reasonably good approximation in the average case, these deviations from the true refraction have been shown to be fairly large both as a function of altitude and of geographical location. The practical importance of these deviations in terms of field-intensity calculations has not as yet been evaluated, but they may turn out to require another type of refraction approximation which allows for the observed variations with respect to height and geographical location.

## Appendix 1. Estimate of Uncertainty in Application of Refractive Index Formula for Moist Air.

There are limitations on the accuracy of both the refractive index formula and the meteorological observations in an individual sounding.

### A. Uncertainty in the Refractive Index Formula.

The formula for the refractive index of moist air, equation (5), page 7, is based on a series of separate laboratory determinations of the dielectric constants of dry air and water vapor[14]. The contributions of each are then combined to yield equation (5),

$$(n-1) \times 10^6 = \frac{79p}{T}(1 + 4800\ e/pT)$$

(a) Dry Air Refractive Index

The refractive index, $n_d$, of dry air is given by:

$$(n_d - 1) \times 10^6 = 79p_d/T \tag{16}$$

where $p_d$ is the partial pressure of dry air. There have been many measurements taken on the dielectric constant of dry air. The most recent determination has been made by Hector and Woernley[14]. From their data the coefficient in equation (16) is computed to be 76.5, whereas the average value obtained by all workers in the field is 77.9. It is pointed out in the same paper that the internal consistency of each worker's data has been better than the agreement from observer to observer. The entire range of the sixteen determinations, the first of which was made by Boltzmann in 1874, is from 72.8 to 80.8. The computed probable error is 4% of the mean. It should be pointed out that the coefficient in equation (16) is about 1% larger than the mean value for all workers in the field. Hence, we take 5% as a reasonable value of the uncertainty in the dry air contribution.

(b) Water Vapor Refractive Index.

The refractive index, $n_w$, of water vapor is given by:

$$(n_w - 1) \times 10^6 = \frac{79e}{T}(1 + 4800/T) \tag{17}$$

According to reference 9, the best determinations over a range of temperature values are those due to Sänger and those due to Stranathan:

$$(n_w - 1) \times 10^6 = \begin{cases} \frac{e}{T}\left(68 + \frac{3.78 \times 10^5}{T}\right) & \text{Sänger} \quad (18) \\[2ex] \frac{e}{T}\left(73 + \frac{3.73 \times 10^5}{T}\right) & \text{Stranathan} \quad (19) \end{cases}$$

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(14) "Dielectric Constants of Eight Gases," by L. G. Hector and D. L. Woernley, Phys. Rev., Vol. 69, Nos. 3, 4, p.101 (2/46)

The value of the water vapor refractive index actually used in obtaining equation (5) is given by equation (17):

$$(n_w - 1) \times 10^6 = \frac{79e}{T}\left(1 + \frac{4800}{T}\right)$$

instead of the mean value of equations (18) and (19), which is:

$$(n_w - 1) \times 10^6 = \frac{79e}{T}\left(0.89 + \frac{4750}{T}\right)$$

Even assuming a temperature of 50°C, the values of $(n_w - 1) \times 10^6$ obtained from equation (17) do not differ from those calculated from equations (18) and (19) by more than 2.5%.

We consider now the uncertainty in the formula of equation (5) for $(n - 1) \times 10^6$ for moist air due to the respective errors in $(n_d - 1) \times 10^6$ for dry air and $(n_w - 1) \times 10^6$ for water vapor. The surface values of these quantities for a typical station, like Washington, D.C., in October is $(n-1) \times 10^6 = 332$ units, which is composed of $(n_d-1) \times 10^6 = 281$ units and $(n_w-1) \times 10^6 = 51$ units. Thus a 5% error in the dry air contribution, and a 2.5% error in the water vapor contribution, lead to a weighted error in the formula for the refractive index of moist air of 4%.

B. Uncertainty in Meteorological Data.

(a) Dry Air Refractive Index

The following error data used in discussing the dry air refractive index has been computed from data in U.S. Weather Bureau memoranda dated May 15, 1943 and May 19, 1943, and is based on Gregg's Standard Atmosphere which represents average U.S. data at 40°N latitude.

Table 1: Probable Error in Computed Pressure and Temperature at Fixed Levels for a Single Sounding for U.S. Weather Stations.

| Fixed Altitude | Pressure mb | % Error in Computed Pressure | Temp. °C | % Error in Computed Temperature | % Error in $(n_d-1) \times 10^6$ |
|---|---|---|---|---|---|
| 10,000 ft | 701.1 | ±0.1 | - 1.4 | ±0.3 | ±0.4 |
| 20,000 ft | 471.8 | ±0.28 | -20.6 | ±0.48 | ±0.8 |
| 10 km | 270.5 | ±0.59 | -44.5 | ±0.66 | ±1.3 |
| 13 km | 171.0 | ±0.82 | -55 | ±0.69 | ±1.5 |
| 16 km | 107.0 | ±0.93 | -55 | ±0.69 | ±1.6 |

The error below altitudes of 20,000 ft ( $\sim$6 km) in $(n-1) \times 10^6$ for moist air, due to the error in the dry air refractive index, must be less than about 0.8%. At 10 km and higher, the error is larger, but there is a negligible water vapor contribution at these altitudes and so the tabulated error then represents the total error. Below about 10 km, we must also consider the effect of the water vapor refractive index contribution, $(n_w-1) \times 10^6$.

(b) Water Vapor Refractive Index

For our purposes the relative humidity error is taken as ±10% RH ( ±10 RH units) at all values of relative humidity. At elevations from 10 km on up, the water vapor contribution is negligible because of the negligible amount of water vapor at these elevations. The water vapor contribution, equation (17), may now be written:

$$(n_w-1) \times 10^6 = \frac{79e_s\ (RH)}{T}\left(1 + \frac{4800}{T}\right) \cong \frac{79e_s\ (RH)\ (4800)}{T^2} \qquad (17)$$

where the last approximation holds at ordinary temperatures. Now, we see in Table 1 that the error in the temperature measurement is of the order of 1% below 10 km. An examination of equation (17) will show that the error in $(n_w-1) \times 10^6$ for water vapor, due to the temperature error, is of the order of 2%. The corresponding error in $e_s$, the saturation specific humidity, due to the 1% error in T, is less than 1%. The error due to the uncertainty in the relative humidity measurement has been calculated for a typical station, taken as Washington, D.C., in October, and tabulated in Table 2, page 15. The total percentage error in $(n_w-1) \times 10^6$ for water vapor, listed in column 3, was obtained by adding a 1% error due to $e_s$, a 2% error due to the effect of the temperature, and the equivalent percentage error due to an absolute error of ±10 RH units for any value of relative humidity. At the surface, the error data are omitted since humidity measurements are made with a sling psychrometer and the accuracy of the data far exceeds that obtainable with the radiosonde humidity element.

We thus see that there is a weighted percentage error of about 3% below 20,000 ft ( $\simeq$ 6 km) in the refractive index computation of moist air due to errors in measurement of the meteorological elements. The 3% includes 2% for the water vapor contribution, and 1% for the dry air contribution.

Our conclusion then is that the total uncertainty in the refractive index calculations is of the order of 7%, about 4% being due to the uncertainty in the coefficients of the formula, and about 3% due to the errors of measurement of the meteorological factors.

Table 2: Error in $(n_w-1) \times 10^6$ and Corresponding Error in $(n-1) \times 10^6$ for Moist Air (Washington, D.C., October)

| Height (km) | Percent Relative Humidity | Percent Error in $(n_w-1)\times10^6$ | $(n_w-1)\times10^6$ | Error in $(n_w-1)\times10^6$ | $(n-1)\times10^6$ for Moist Air | Percent Error in $(n-1)\times10^6$ for Moist Air, due to Water Vapor |
|---|---|---|---|---|---|---|
| 0 | 75 | — | 51 | — | 332 | — |
| 1 | 65 | ±18 | 40 | ±7 | 291 | ±2 |
| 2 | 60 | ±20 | 29 | ±6 | 255 | ±2 |
| 3 | 52 | ±22 | 18 | ±4 | 221 | ±2 |
| 4 | 49 | ±23 | 12 | ±3 | 195 | ±2 |
| 5 | 48 | ±24 | 8 | ±2 | 172 | ±1 |
| 6 | 48 | ±24 | 5 | ±1 | 152 | ±1 |

## Appendix 2. Extrapolation of Refractive Index Data to 30 km

The expression for the pressure in an isothermal atmosphere is given by:

$$p = p_o e^{-(h-h_o)/R'T_s} \tag{14}$$

where: $T_s$ is the constant temperature of the stratosphere in $^oK$, but which may vary with season.

$p_o$ is the surface pressure in mb which in this case refers to the base of the stratosphere, but the actual value is unimportant for the present application as shown below.

$R'$ is the gas constant for the dry stratosphere.

$h-h_o$ is the height referred to the base of the stratosphere.

Assuming no water vapor at these elevations, the refractive index is given by:

$$(n-1) \times 10^6 = 79p/T_s = \frac{79p_o}{T_s} e^{-(h-h_o)/R'T_s}$$

and hence, $\log(n-1) = k_1 - k_2 h$ (15)

where $k_1$ and $k_2$ are constants. Thus by plotting the stratosphere data that we have for the location concerned (see Figure 11, page 34), we should get a straight line which allows us to extrapolate refractive index data to great heights. In fact, we find that (n-1) reduces to $4 \times 10^{-6}$ units at an elevation of 30 km at Washington, D.C. in October, if we assume the stratosphere to be isothermal to that elevation.(11)

## Appendix 3: Derivation of Formula for Refraction, $\tau$, of a Radio Ray.

Snell's law for spherical, refracting surfaces is: $nr \cos \theta = \text{constant}$ The expression for curvature at a point on the ray path, to be derived below from equation (7), is:

$$\frac{d\tau}{ds} = \frac{1}{\rho} = -\frac{1}{n}\frac{dn}{dh}\cos\theta \tag{20}$$

where: $\rho$ is the radius of ray curvature at a point on the ray.

$(\frac{dn}{dh})$ is the vertical gradient of refractive index at the point in question.

$(\frac{d\tau}{ds})$ is the rate of change of curvature at any point along the path.

Differentiating equation (7) (shown above), with respect to height, h, remembering that $r = a + h$, we have:

$$\frac{1}{n}\frac{dn}{dh} + \frac{1}{r} - \tan\theta\frac{d\theta}{dh} = 0$$

$$\text{and} \quad \frac{1}{r}\cos\theta - \sin\theta\frac{d\theta}{dh} = -\frac{1}{n}\frac{dn}{dh}\cos\theta \tag{21}$$

From the infinitesimal section of the ray path in Figure 12, page 35, we have:

$$ds^2 = dh^2 + r^2\,d\phi^2$$

$$\text{and} \quad r\frac{d\phi}{ds} = \cos\theta$$

$$\text{and} \quad \frac{dh}{ds} = \sin\theta$$

and, substituting in equation (21), for $\cos\theta$ and $\sin\theta$, we have:

$$\frac{d(\phi-\theta)}{ds} = -\frac{1}{n}\frac{dn}{dh}\cos\theta \tag{22}$$

From geometrical considerations for the quadrilateral indicated by dashed lines in Figure 12,

$$(\frac{\pi}{2} + \theta) + (\pi - \Delta\tau) + \Delta\phi + \left[\frac{\pi}{2} - (\theta + \Delta\theta)\right] = 2\pi$$

and hence, $\Delta\tau = \Delta\phi - \Delta\theta$ and, $\dfrac{\Delta\tau}{\Delta s} = \dfrac{\Delta(\phi - \theta)}{\Delta s}$

Passing to the limit as $\Delta s \to 0$, we have, substituting in equation (22),

$$\frac{d\tau}{ds} = -\frac{1}{n}\frac{dn}{dh}\cos\theta \qquad (20)$$

Integrating between the altitudes $h_1$ and $h_2$,

$$\tau_{1,2} = -\int_{n_1}^{n_2} \frac{1}{n}\frac{dn}{dh}\cos\theta \, ds = -\int_{n_1}^{n_2} \frac{1}{n}\cot\theta \, dn$$

and since $n \cong 1$, $\tau_{1,2} = -\int_{n_1}^{n_2} \cot\theta \, dn$

Now, $\frac{dn}{dh} = \frac{d(\Delta n)}{dh}$ and changing the variable of integration to $\Delta n$, by means of the relation $n = 1 + \Delta n$

we have $$\tau_{1,2} = -\int_{\Delta n_1}^{\Delta n_2} \cot\theta \, d(\Delta n)$$

By the definition of the mean value of $\cot\theta$ over the interval $(\Delta n_1, \Delta n_2)$,

$$\overline{(\cot\theta)}_{1,2} = \frac{\int_{\Delta n_1}^{\Delta n_2} \cot\theta \, d(\Delta n)}{(\Delta n_2 - \Delta n_1)} \qquad (23)$$

and, therefore, we have finally,

$$\tau_{1,2} = \overline{(\cot\theta)}_{1,2}\,(\Delta n_1 - \Delta n_2) \qquad (6)$$

## Appendix 4. Applications of Formula for Refraction of a Radio Ray.

To determine the smallest number and, therefore, the largest size of the height intervals needed, at angles $\theta_0 \leqq 10^\circ$, consistent with the 7% error arising from the computation of $\Delta n$ as shown in Appendix 1, page 12, we consider the following facts:

Under the condition that $\frac{dn}{dh} = \frac{\Delta n - \Delta n_1}{h - h_1}$ is constant in the interval $(h_2, h_1)$ we have,

$$\overline{(\cot\theta)}_{1,2} = \frac{\int_{\Delta n_1}^{\Delta n_2} \cot\theta \, d(\Delta n)}{\Delta n_2 - \Delta n_1} = \frac{\int_{h_1}^{h_2} \cot\theta \, dh}{h_2 - h_1}$$

Now, under the same condition, Snell's law for low angles of arrival in the interval becomes,

$$\frac{\theta^2 - \theta_1^{\,2}}{2} = \left(\Delta n + \frac{h}{a}\right) - \left(\Delta n_1 + \frac{h_1}{a}\right) = \left(\frac{\Delta n - \Delta n_1}{h - h_1} + \frac{1}{a}\right)(h - h_1)$$

$$= \frac{h - h_1}{ka} \tag{24}$$

where $\frac{1}{ka} = \frac{1}{a} - \frac{1}{\rho}$, $\rho$ being the radius of curvature of the ray. This shows $\theta^2$ to be a linear function of h over the interval. We therefore plot $\theta^2$ against h, to examine how wide are the intervals over which this linearity is indicated. An example of this relationship is shown in Figure 13, page 36, for the case $\theta_0 = 0$ for Washington, D.C., in October. This empirical relation makes it possible to introduce a great simplification in determining the value of $\overline{(\cot\theta)}$ over such an interval. For, at these low angles $\cot\theta \cong 1/\theta$, and it may be shown that $\overline{\cot\theta} = 1/\theta_m$, where $\theta_m$ is the arithmetical mean of $\theta$ on the interval, e.g., over the interval $(h_1, h_2)$, $\theta_m = (\theta_1 + \theta_2)/2$. From Snell's law $nr\cos\theta$ = constant, we have:

$$\cot\theta = ka\frac{d\theta}{dh}$$

Thus,

$$\overline{\cot\theta} = \frac{\int_{h_1}^{h_2} \cot\theta \, dh}{h_2 - h_1} = ka\,\frac{\theta_2 - \theta_1}{h_2 - h_1}$$

and therefore, for small angles, $\overline{\cot\theta} = \frac{2}{\theta_1 + \theta_2} \cong \frac{1}{\theta_m}$.

When this value of $\cot\theta$ is substituted in equation 6, we obtain, simply,

$$\tau_{1,2} = \frac{\Delta n_1 - \Delta n_2}{\theta_m} \qquad \text{in radians} \quad 0 \leqq \theta_0 \leqq 10^\circ$$

In the example shown in Figure 13, we see that $\theta^2$ can be considered <u>arbitrarily</u> to be a linear function of h over four separate intervals from

the ground to 18 km, namely, (1) surface to 0.5 km; (2) 0.5 to 2.5 km; (3) 2.5 to 6.0 km; and (4) 6 to 18 km. Above 18 km, the contribution to the total refraction is small and may be shown to be given approximately by $\tau = \Delta n_{18} \cot \theta_{18}$, where the subscripts refer to the values of the quantities at 18 km. In general, the same intervals may be used at all locations, since the shape of the $\theta^2$ vs h curve does not change appreciably from location to location.

We show at this point a simple formula for the refraction of a radio ray tangential to the earth's surface in an atmospheric layer in which an effective earth's radius factor k may be used. Equation (24) shows that in this case,

$$\frac{\theta^2}{2} = \frac{h}{ka}$$

In addition the well-known distance-height formula for a ray tangent to the earth's surface in an atmosphere with effective earth's radius factor k is:

$$d = \sqrt{2ka\,h}$$

Substituting these relations in equation (12), page 10, we have,

$$\tau_{1,2} = \frac{d_{1,2}}{a} + (\theta_1 - \theta_2) \tag{12}$$

and

$$\tau = \sqrt{\frac{2h}{a}}\left(\sqrt{k} - \frac{1}{\sqrt{k}}\right), \qquad \theta_1 \text{ being } 0.$$

At large angles, $\theta_o$ 10°, we now derive the following expression for the total refraction,

$$\tau = \Delta n_o \cot \theta_o \qquad \text{in radians} \tag{11}$$

the approximation being implied that $\overline{(\cot \theta)} \cong \cot \theta_o$, in equation (6)

$$\tau_{1,2} = \overline{(\cot \theta)}_{1,2}\,(\Delta n_1 - \Delta n_2) \tag{6}$$

We use the subscript o for the ground value, the subscript 18 at h = 18 km, and the subscript $\infty$ for h = $\infty$. Equation (6) may then be transformed to:

$$\tau_{o,\infty} \equiv \tau = \overline{(\cot \theta)}_{0,18}\,(\Delta n_o - \Delta n_{18}) + \overline{(\cot \theta)}_{18,\infty}\,(\Delta n_{18} - 0) \tag{25}$$

or

$$\tau = \overline{(\cot \theta)}_{0,18}\,\Delta n_o - \Delta n_{18}\left[\overline{(\cot \theta)}_{0,18} - \overline{(\cot \theta)}_{18,\infty}\right] \tag{25a}$$

For average refractive index distributions, i.e., there are no height intervals where the downward ray curvature exceeds earth curvature and neglecting ionospheric effects, then as h increases, $\Delta n$ approaches zero and $\theta$ becomes uniformly larger because of the geometry of the ray path. This means that cot $\theta$ decreases uniformly with height, and hence, the average value $\overline{(\cot\theta)}_{18,\infty}$ is less than $\overline{(\cot\theta)}_{o,18}$. Now, $\Delta n_{18}$ is less than $\Delta n_o/10$ at all geographical locations and for all seasons.

Also, for angles $\theta_o = 10^\circ$ or higher, $\dfrac{\overline{(\cot\theta)}_{18,\infty}}{\overline{(\cot\theta)}_{o,18}} \lesssim \dfrac{1}{2}$

Therefore, at most the second term in equation (23a) is 5% of the first term.

In addition, from consideration of data at San Juan, Puerto Rico, in July, using the large-angle formula (eq.(11)) for $\theta = 10^\circ$ which is the case of largest refraction considered in this report, the largest error we make by taking $\overline{(\cot\theta)}_{o,18} \cong \cot\theta_o$ is 4%. Thus, at $\theta_o = 10^\circ$, the error made using the high-angle formula is of the order of 9%, with the error decreasing rapidly as $\theta_o$ increases from $10^\circ$.

Table 3 shows a sample computation using the ray tracing method of this report.

Table 3: Sample Calculation of Refraction - Washington, D.C. (October) $\theta_o = 0_{mr}$

| (1) h (km) | (2) $h/a \times 10^6$ | (3) $\Delta n \times 10^6$ | (4) M | (5) $(M_o - M) = \theta_i^2/2$ | (6) $\theta_i^2$ $(mr)^2$ | (7) $\theta_i$ mr | (8) $\frac{\theta_{i-1} + \theta_i}{2}$ mr | (9) $\Delta n_{i-1} - \Delta n_i$ $\times 10^6$ | (10) = $\frac{(9)}{(8)}$ $\Delta \tau_i$ mr | (11) $\Sigma \Delta \tau_i$ mr |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.025(surface) | 4 | 332 | 336 | 0 | 0 | 0 | - | - | - | - |
| 0.5 | 79 | 310 | 389 | 53 | 106 | 10.3 | 5.2 | 22 | 4.2 | 4.2 |
| 2.5 | 393 | 239 | 632 | 296 | 592 | 24.3 | 17.3 | 71 | 4.1 | 8.3 |
| 6.0 | 942 | 152 | 1094 | 758 | 1516 | 38.9 | 31.6 | 87 | 2.8 | 11.1 |
| 18.0 | 2827 | 30 | 2857 | 2521 | 5042 | 71.0 | 55.0 | 122 | 2.2 | 13.3 |
| 18.0 | | | | | | | 71.0 | 30 | 0.42 | 13.7 |

where $M = (\Delta n + h/a) \times 10^6$ is the only symbol not previously defined and is the so-called M-value, or <u>excess</u> <u>modified</u> <u>refractive</u> <u>index</u>, in millionths.

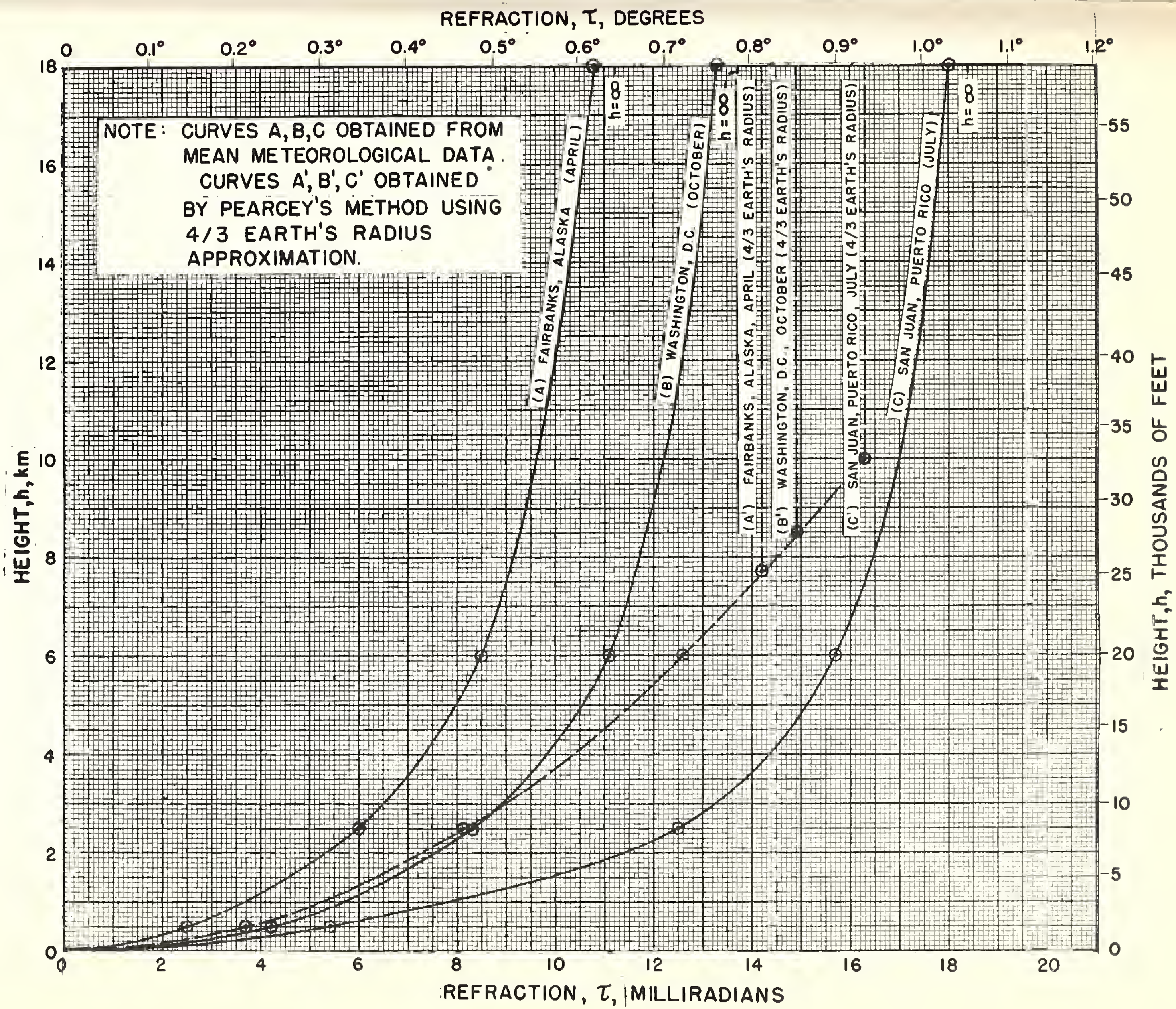



Fig. 1 BENDING BETWEEN GROUND AND HEIGHT, h, OF A RADIO RAY TANGENTIAL WITH EARTH'S SURFACE.

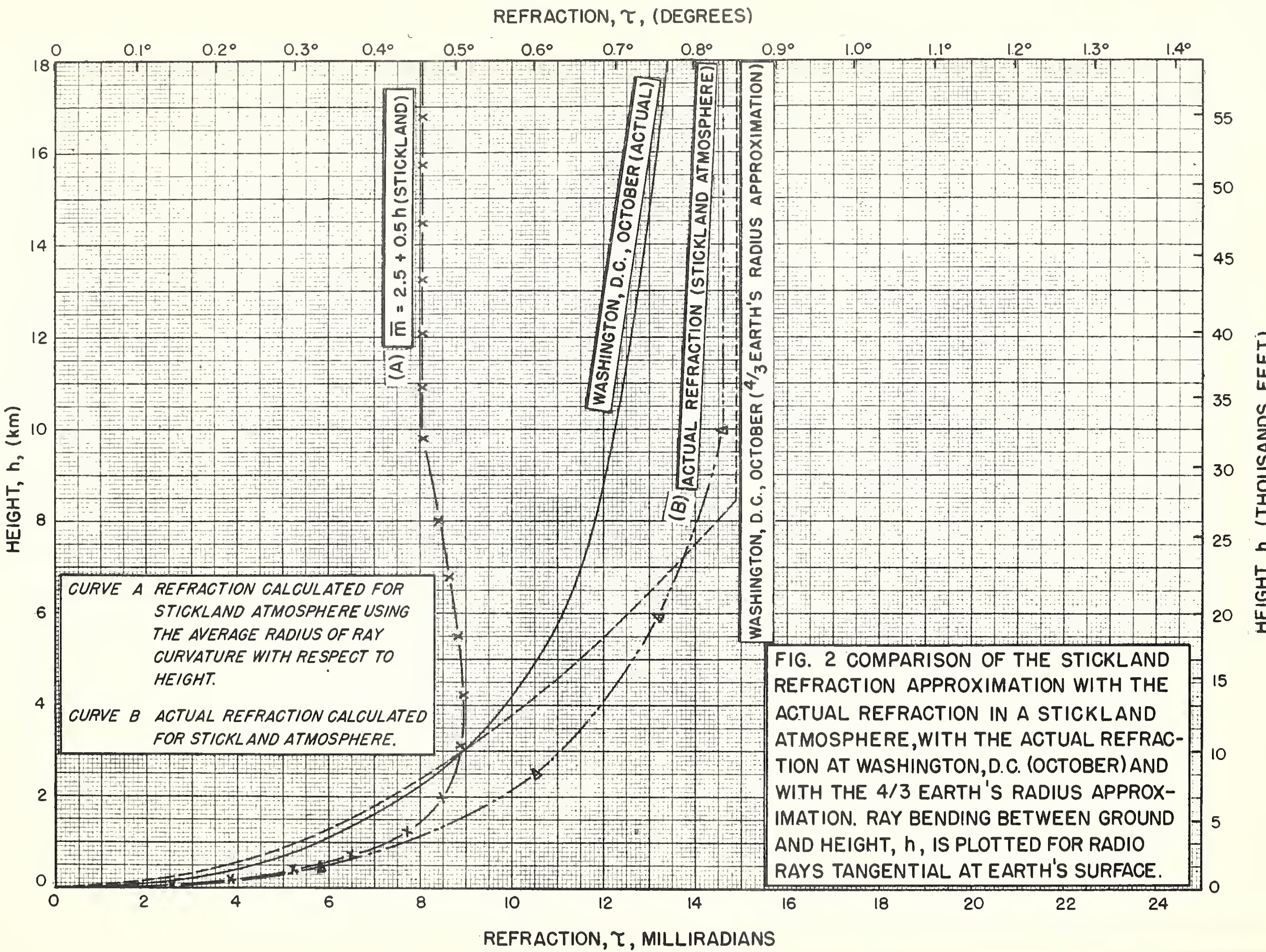



FIG. 2 COMPARISON OF THE STICKLAND REFRACTION APPROXIMATION WITH THE ACTUAL REFRACTION IN A STICKLAND ATMOSPHERE, WITH THE ACTUAL REFRACTION AT WASHINGTON, D.C. (OCTOBER) AND WITH THE 4/3 EARTH'S RADIUS APPROXIMATION. RAY BENDING BETWEEN GROUND AND HEIGHT, h, IS PLOTTED FOR RADIO RAYS TANGENTIAL AT EARTH'S SURFACE.

Fig. 3 PRESSURE-TEMPERATURE TERM FOR THE REFRACTIVE INDEX COMPUTATION
(NUMBERS ON SLANT LINES ARE VALUES OF $79\frac{p}{T}$)

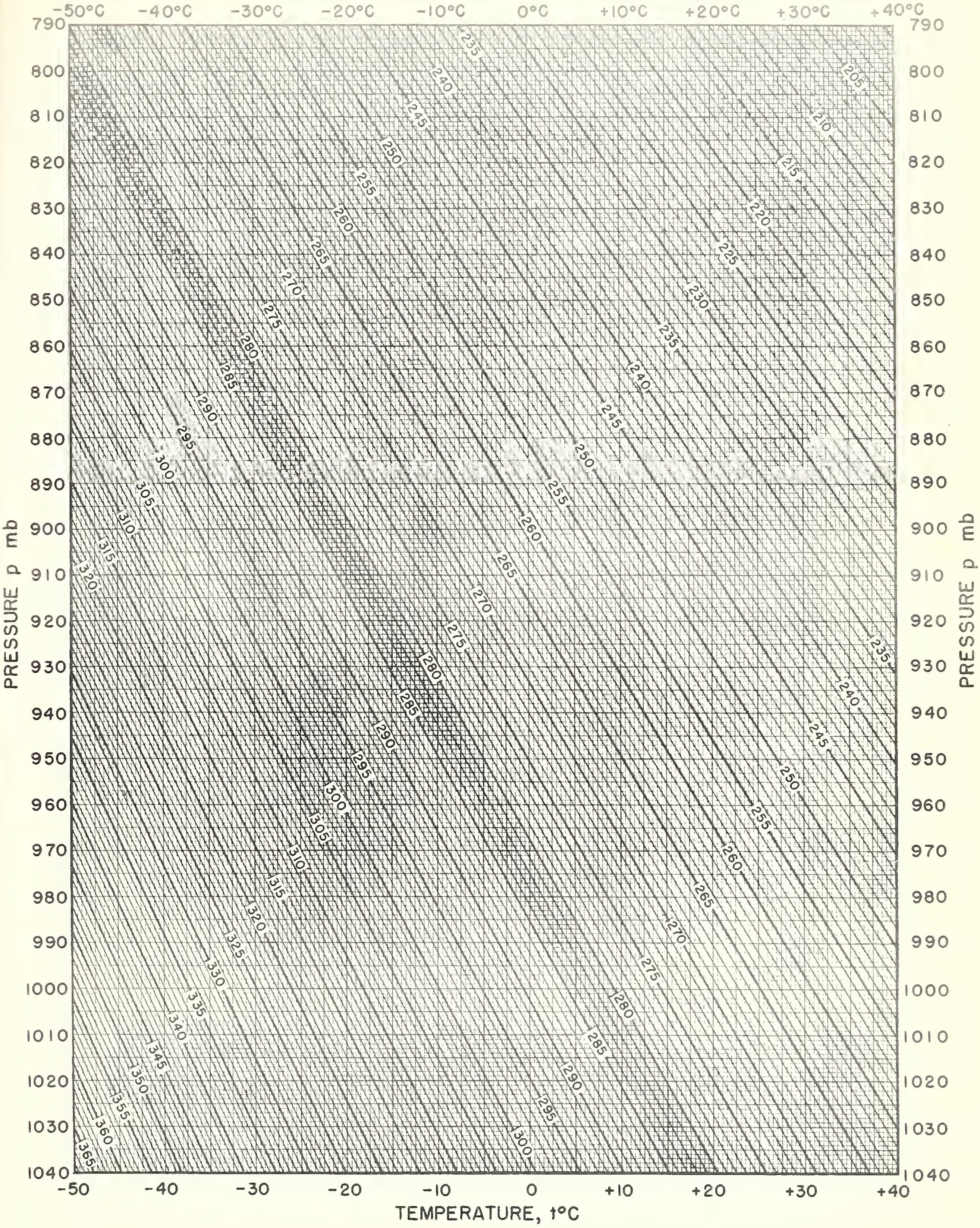


Fig. 4 PRESSURE-TEMPERATURE TERM FOR THE REFRACTIVE INDEX COMPUTATION
(NUMBERS ON SLANT LINES ARE VALUES OF $79\frac{p}{T}$)

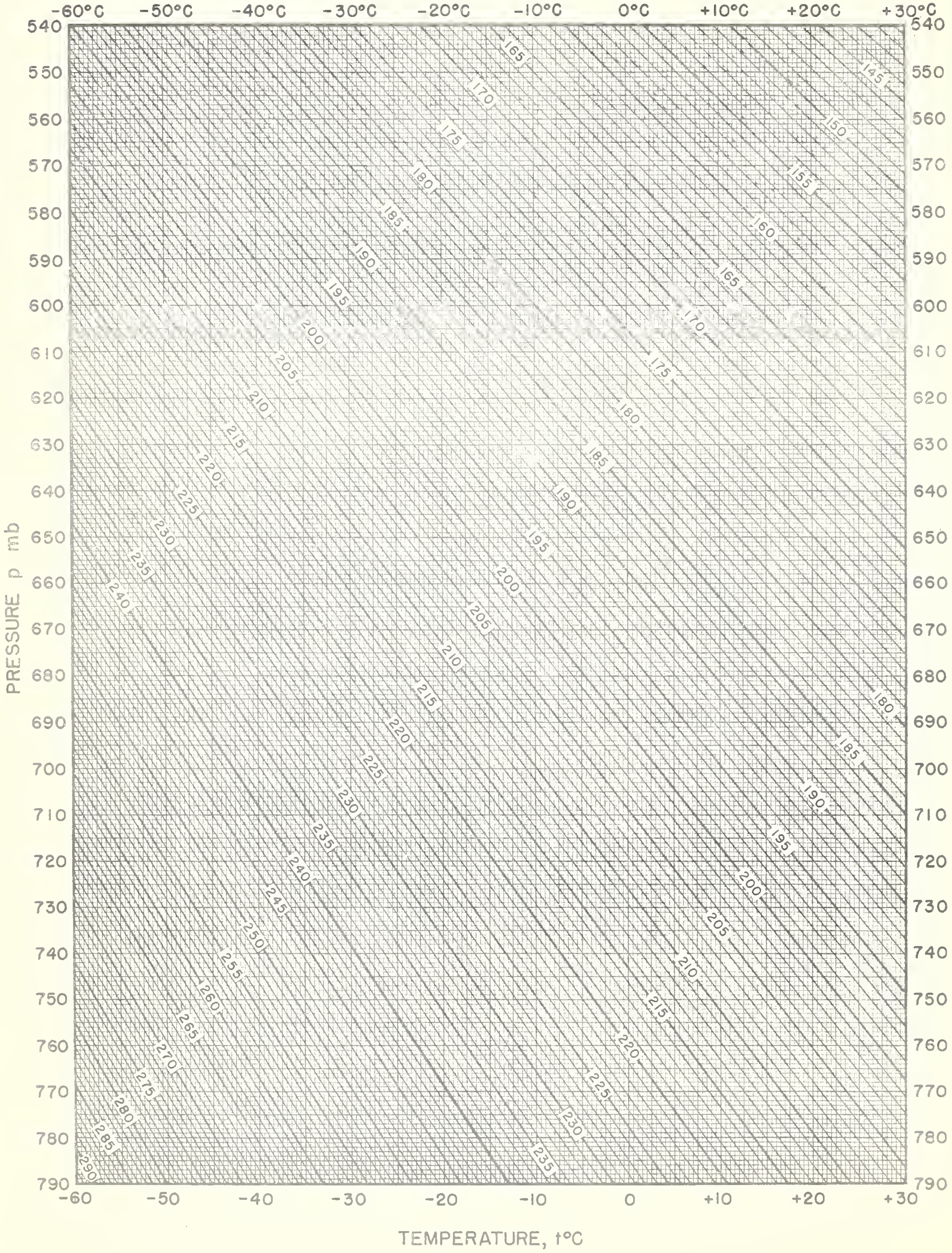


Fig. 5 PRESSURE-TEMPERATURE TERM FOR THE REFRACTIVE INDEX COMPUTATION (NUMBERS ON SLANT LINES ARE VALUES OF $79\frac{p}{T}$)

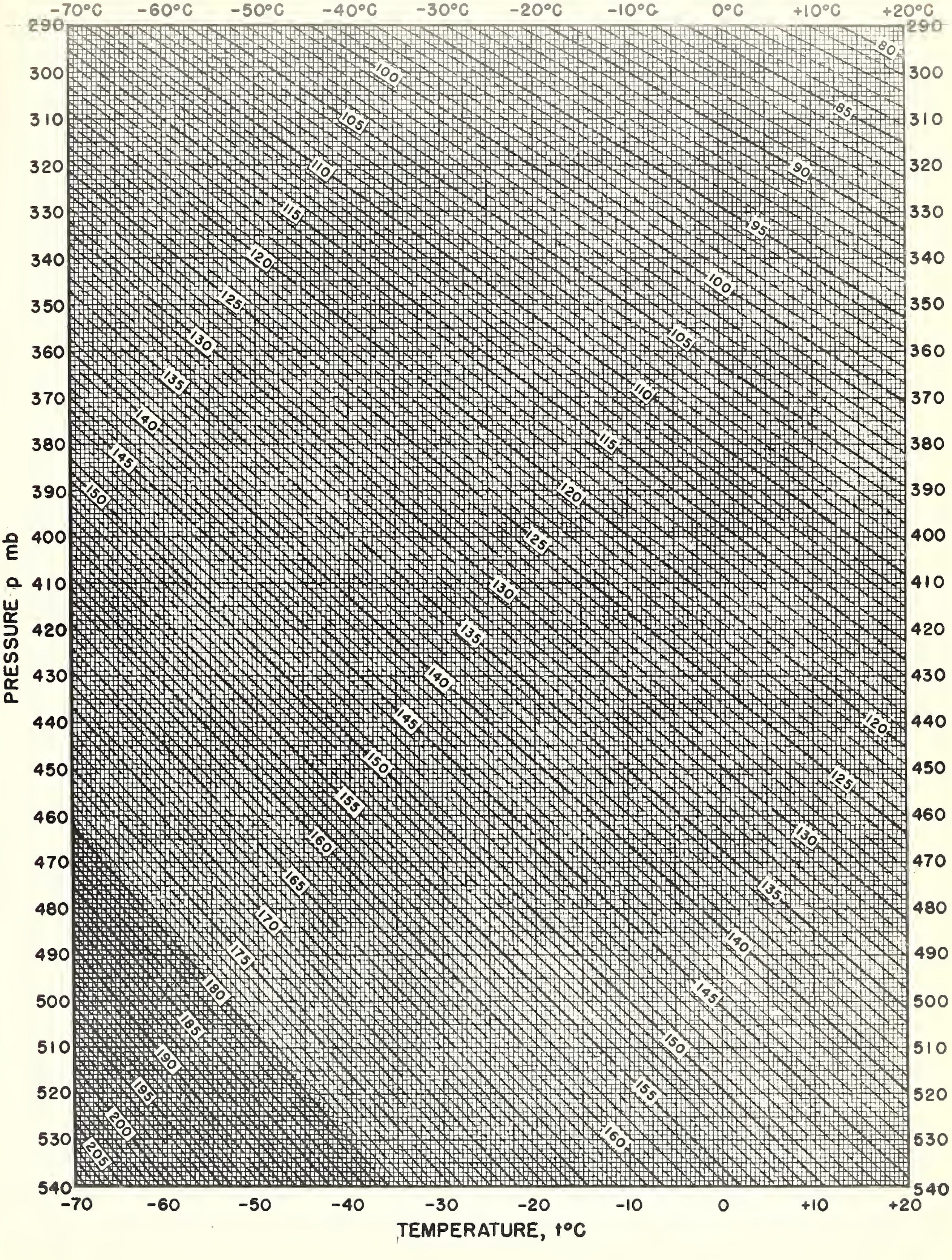
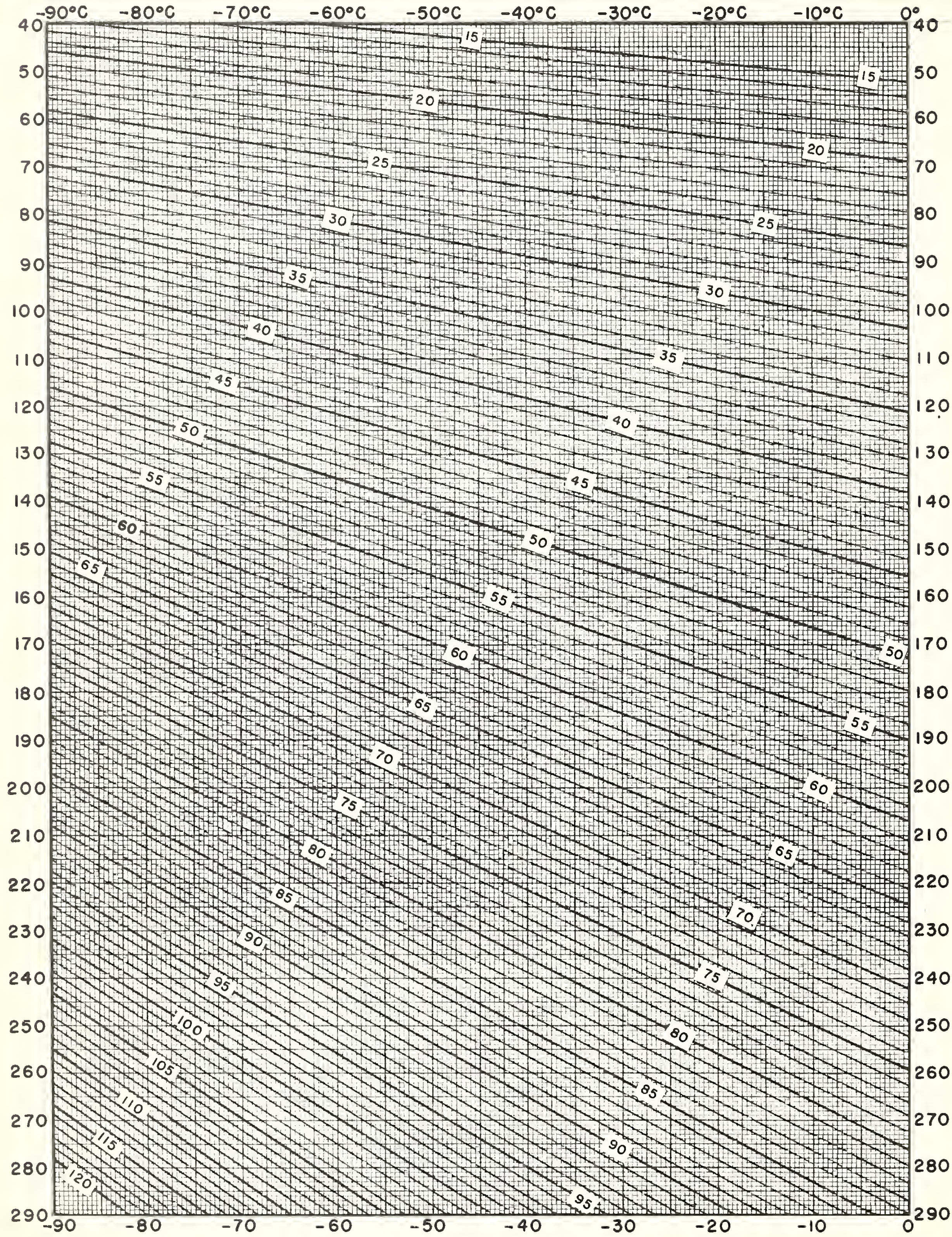


Fig. 6 PRESSURE-TEMPERATURE TERM FOR THE REFRACTIVE INDEX COMPUTATION
(NUMBERS ON SLANT LINES ARE VALUES OF $79\frac{p}{T}$)



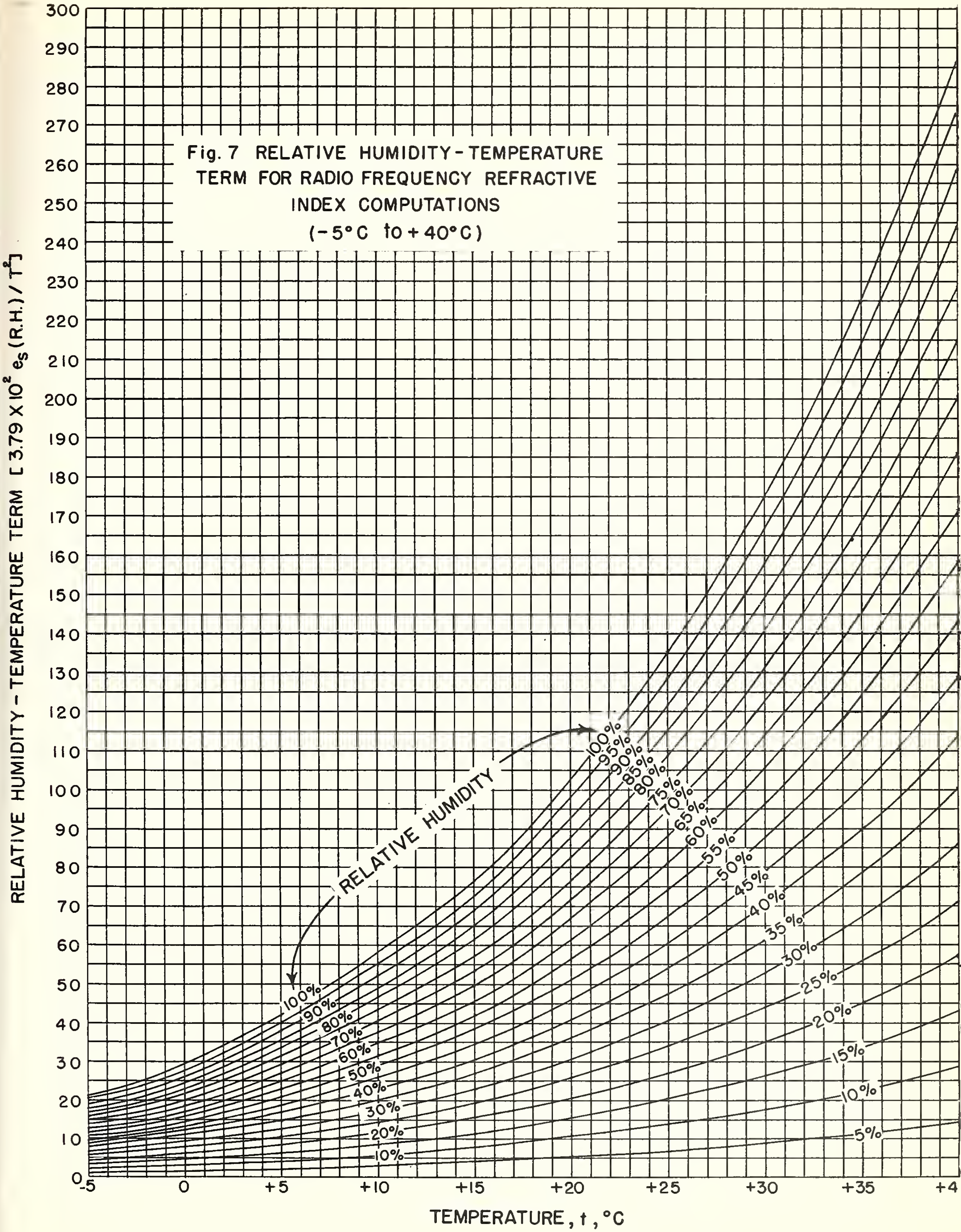


Fig. 7 RELATIVE HUMIDITY - TEMPERATURE TERM FOR RADIO FREQUENCY REFRACTIVE INDEX COMPUTATIONS (-5°C to +40°C)
RELATIVE HUMIDITY - TEMPERATURE TERM [3.79 X 10² e_s (R.H.) / T²]
TEMPERATURE, t, °C
RELATIVE HUMIDITY
100%
95%
90%
85%
80%
75%
70%
65%
60%
55%
50%
45%
40%
35%
30%
25%
20%
15%
10%
5%

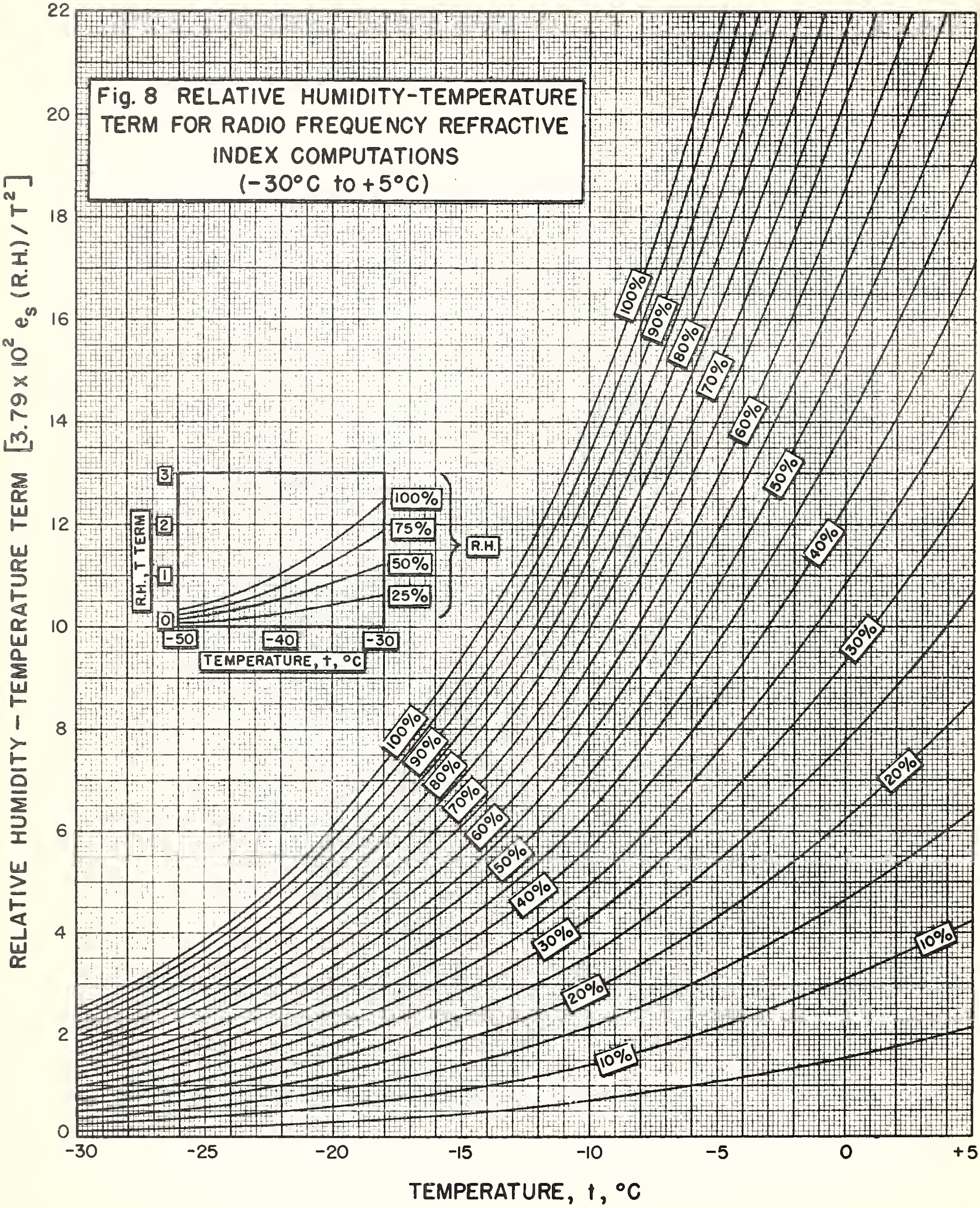


Fig. 8 RELATIVE HUMIDITY-TEMPERATURE TERM FOR RADIO FREQUENCY REFRACTIVE INDEX COMPUTATIONS (-30°C to +5°C)
RELATIVE HUMIDITY - TEMPERATURE TERM [3.79 x 10² e_s (R.H.) / T²]
TEMPERATURE, t, °C
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
R.H., T TERM
R.H.
75%
25%
-50
-40
-30
-25
-20
-15
-10
-5
0
+5
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2
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8
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12
14
16
18
20
22

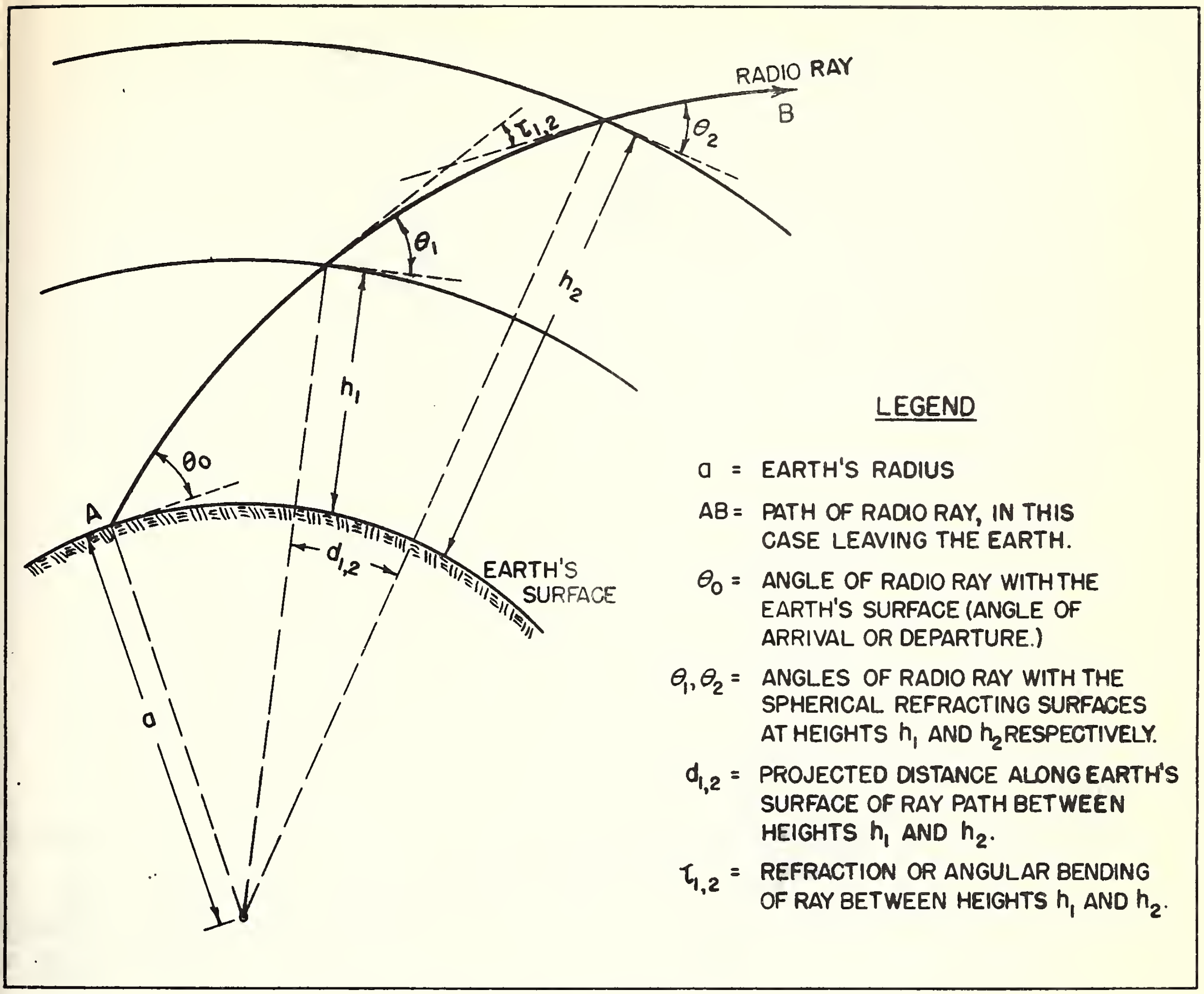



Fig. 9 REFRACTION $\tau_{1,2}$ IN THE EARTH'S ATMOSPHERE BETWEEN HEIGHTS $h_1$ AND $h_2$

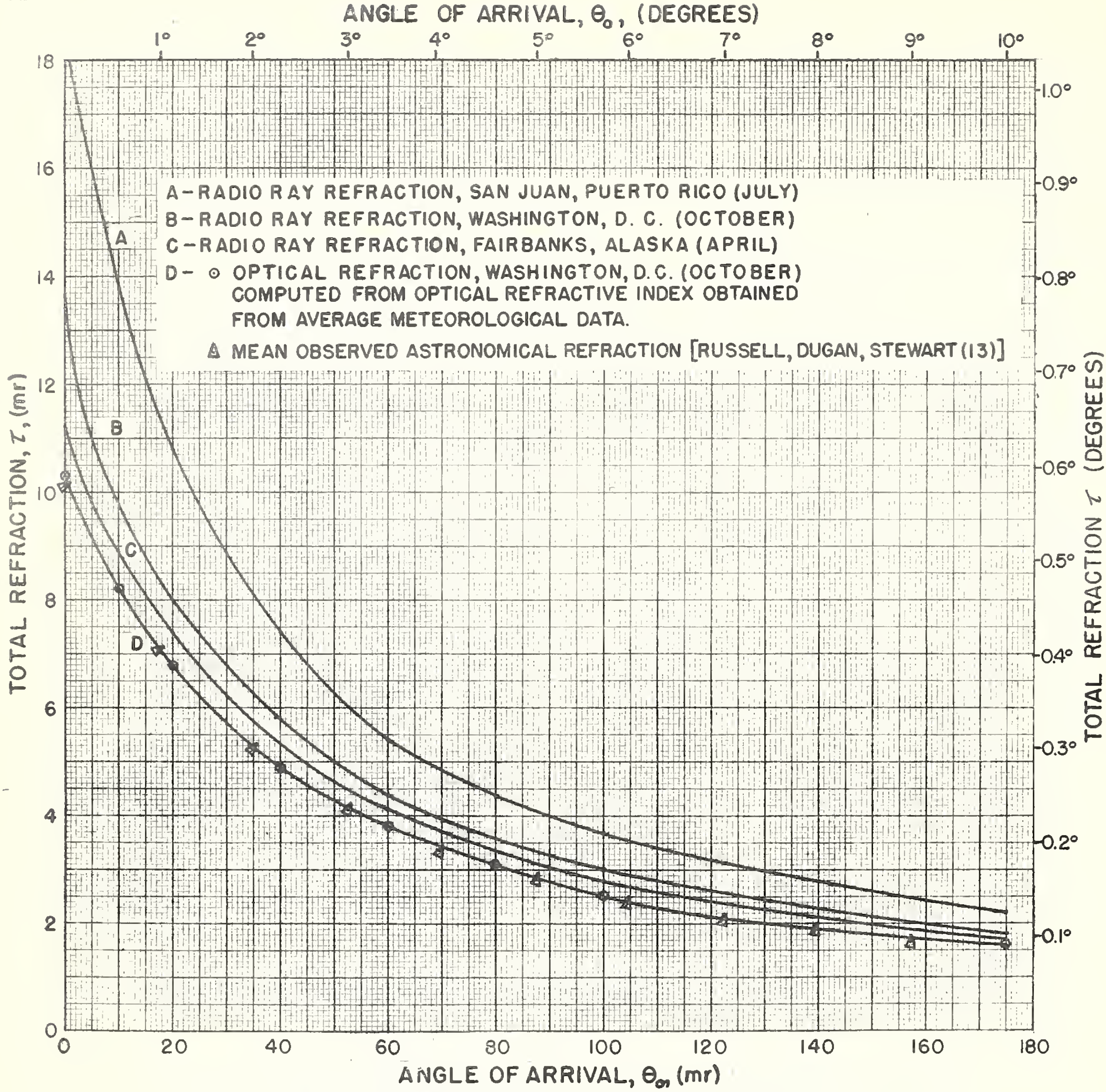



Fig. 10 TOTAL REFRACTION VS ANGLE OF ARRIVAL, $\theta_0 \leq 10°$

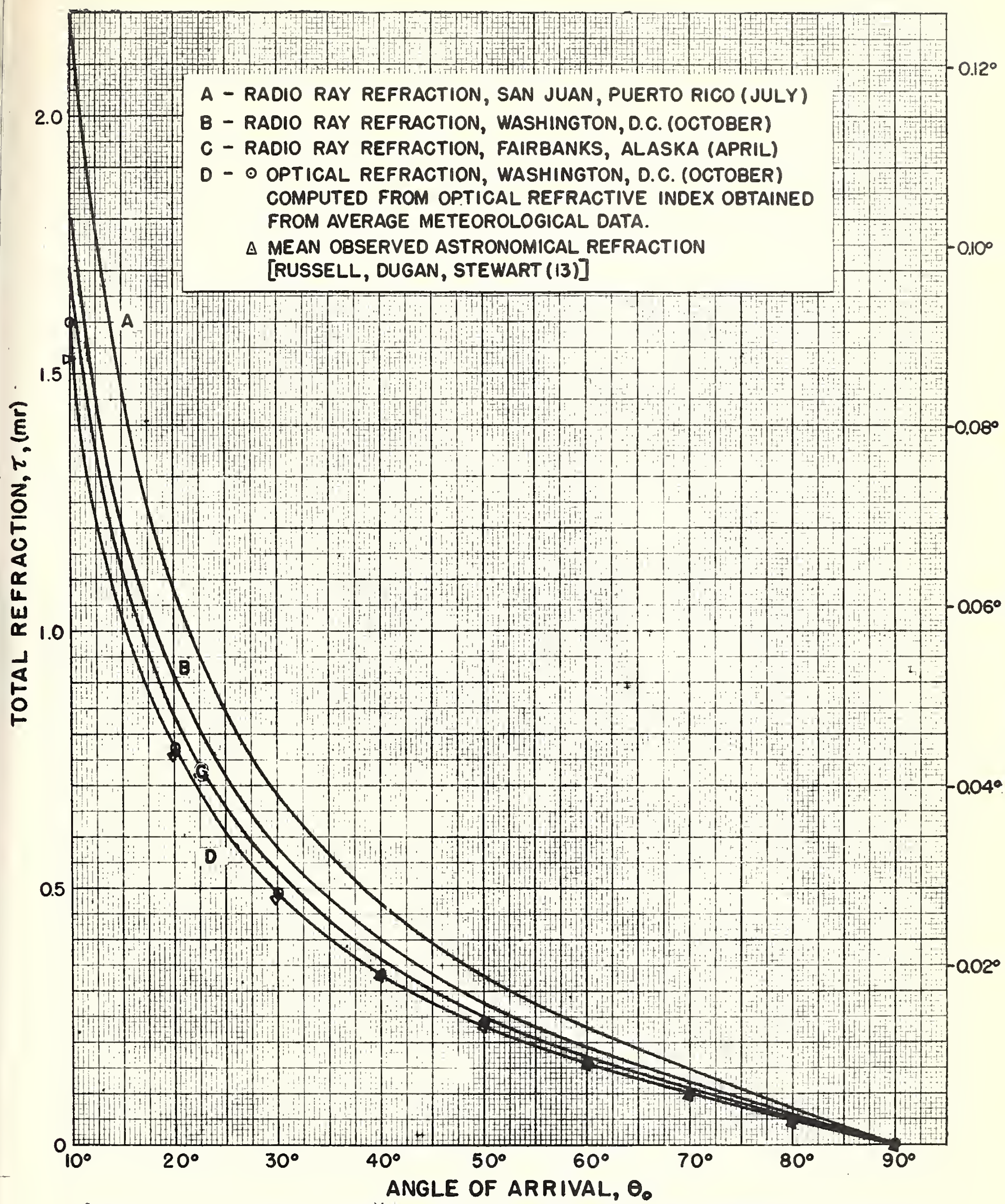
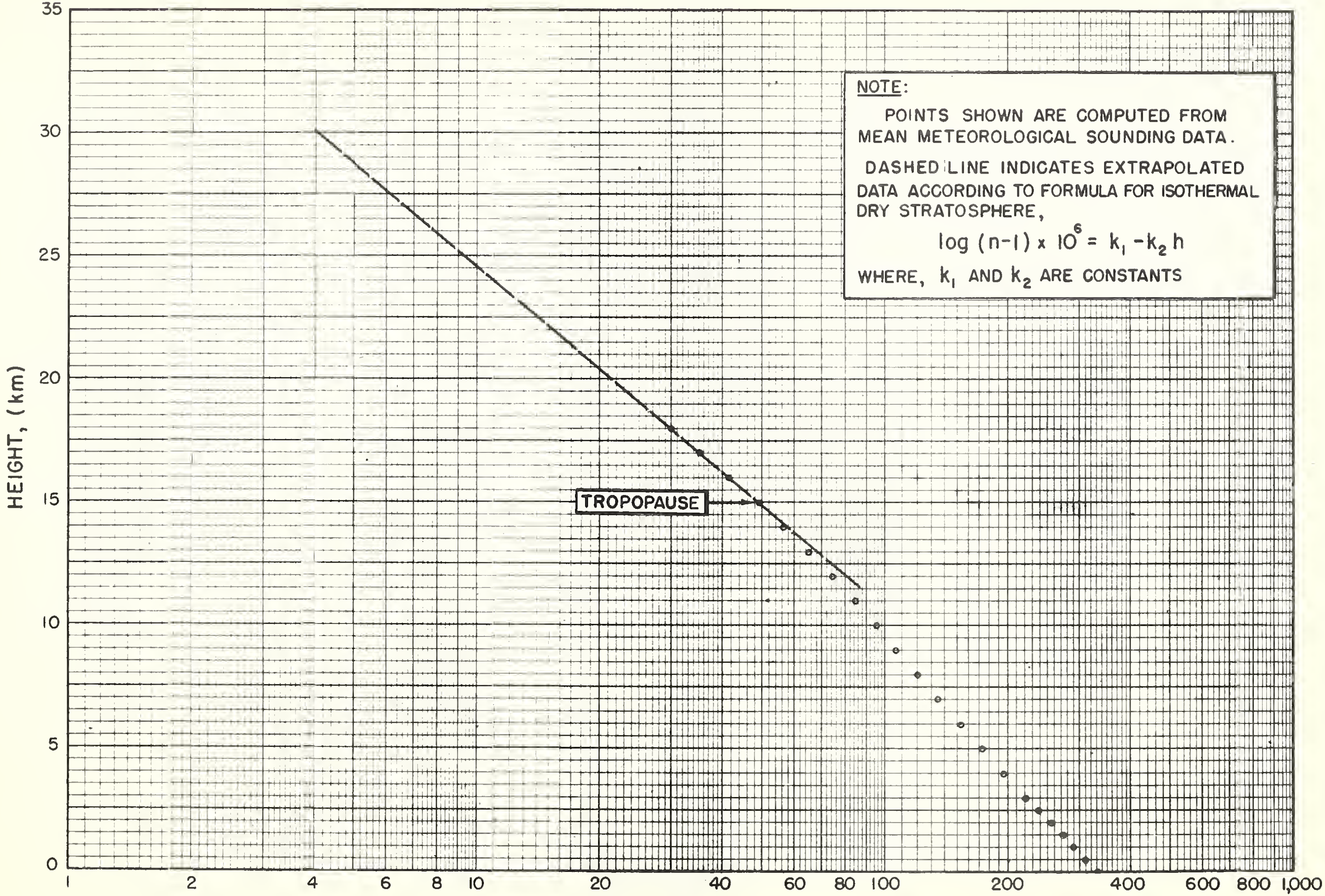



Fig. 10a TOTAL REFRACTION VS ANGLE OF ARRIVAL, $\theta_0 \geq 10°$

Fig. 11 REFRACTIVE INDEX DISTRIBUTION IN THE TROPOSPHERE AND STRATOSPHERE (WASHINGTON, D.C., OCTOBER)



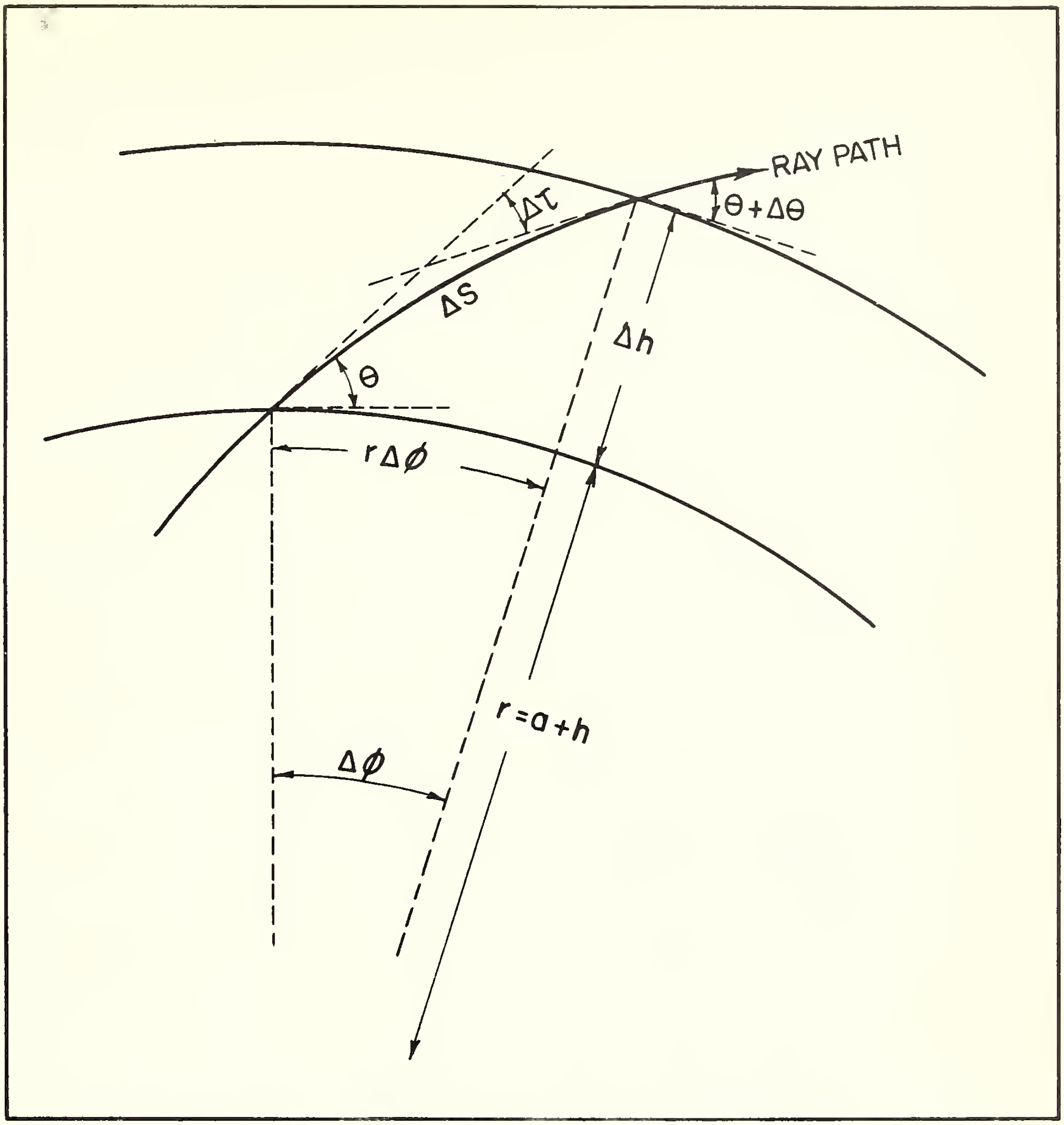



Fig. 12 INFINITESIMAL SECTION OF RAY PATH

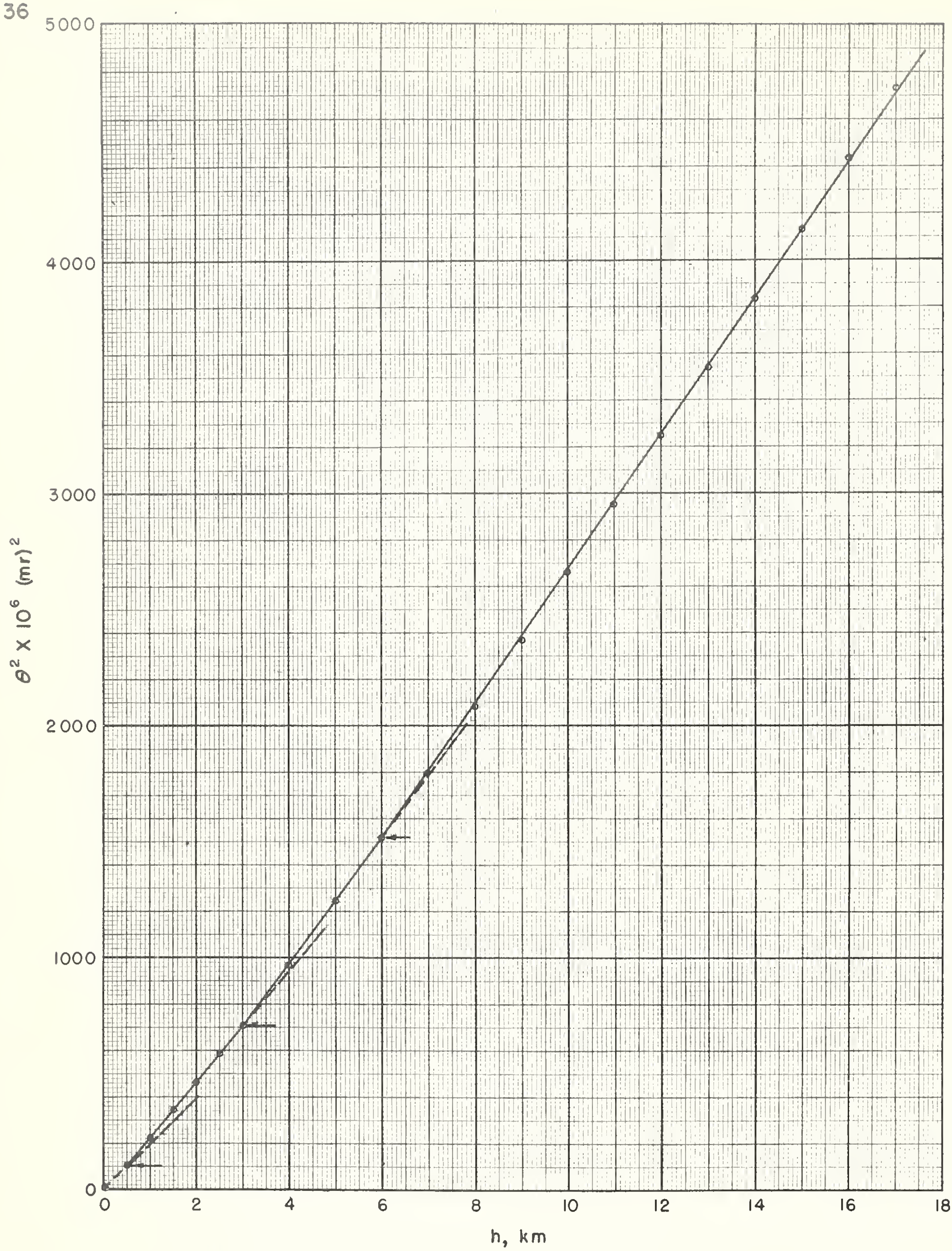



Fig. 13 $\theta^2$ VS. HEIGHT ($\theta$ IS THE ANGLE OF ARRIVAL AT HEIGHT h) WASHINGTON, D. C., OCTOBER, $\theta_0$ = 0 mr.

$$\theta^2 = \theta_0^2 + 2\left(\Delta n + \frac{h}{a}\right) - 2\left(\Delta n_0 + \frac{h_0}{a}\right)$$